

©Journal of Technical University at Plovdiv
 Fundamental Sciences and Applications, Vol. 2, 1996
Series A-Pure and Applied Mathematics
 Bulgaria, ISSN 1310-8271

Note on the multipliers of Cauchy integrals of logarithmic potentials

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Abstract

The present note contains a generalization of a theorem of Hallenbeck and Samotij for the multipliers of Cauchy integrals of logarithmic potentials.

1 Introduction.

Let D denote the unit disk in the complex plane and T the unit circle. Let M be the Banach space of all complex-valued Borel measures on T with the usual variation norm. For $\alpha \geq 0$, let F_α denote the family of the analytic functions g , for which there exists $\mu \in M$ such that

$$g(z) = \int_T \frac{1}{(1 - \bar{\zeta}z)^\alpha} d\mu(\zeta) = K_\alpha^\mu, \quad \alpha > 0,$$

$$g(z) = g(0) + \int_T \log \left(\frac{1}{1 - \bar{\zeta}z} \right) d\mu(\zeta) = K_0^\mu, \quad \alpha = 0.$$

We note that F_α is a Banach space with the natural norm

$$\|g\|_{F_\alpha} = \inf \{ \|\mu\| : \mu \in M, \quad g = K_\alpha^\mu \}.$$

Let m_α denote the set of all multipliers of F_α and

$$\|f\|_{m_\alpha} = \sup \{ \|fg\|_{F_\alpha} : \|g\|_{F_\alpha} \leq 1 \}.$$

⁰**1991 Mathematics Subject Classification:**Primary 30E20, 30D50

⁰**Key words and phrases:**Analytic function, Cauchy integrals, multipliers.

⁰*Received June 15, 1996.*

The following results were proved in [1].

Theorem A. $m_0 \subset m_\alpha \subset H^\infty$ ($m_0 \neq m_\alpha$) for each $\alpha > 0$.

Theorem B. If f is an analytic function on D and $f' \in H^p$ for some $p > 1$, then $f \in m_0$.

Theorem C. For every $p \geq 1$ there is a constant c_p , such that for every $f \in m_0$ and each $\varsigma \in T$ we have

$$\int_0^1 \left((1-r) \log \frac{1}{1-r} \right)^{p-1} |f'(r\varsigma)|^p dr \leq c_p \|f\|_{m_0}.$$

The present note contains a generalization of **Theorem C**.

2 Main result

Theorem D. If $f \in m_0$ then for each $\varsigma \in T$ and $0 \leq r < 1$ we have

$$a) |f'(r\varsigma)| (1-r) \log \frac{1}{1-r} \leq \|f\|_{m_0} + \|f\|_{H^\infty} < \infty ;$$

$$b) \int_0^1 |f'(r\varsigma)| dr \leq \frac{\pi}{2} \|f\|_{m_1} < \infty.$$

Proof. a) Let $f \in m_0$, $g_0 = \log \frac{1}{1-z}$ and $\xi = r\varsigma \in D$.

By Closed Graph Theorem the multiplication operator $F_0 \ni g \rightarrow fg \in F_0$ is bounded and $\|fg\|_{F_0} \leq \|f\|_{m_0} \|g\|_{F_0}$.

Since $g_0(z \bar{\xi} / |\xi|) \in F_0$ and $\|g_0(z \bar{\xi} / |\xi|)\|_{F_0} = 1$ for each $\xi \in D$, then

$$\|f(z) g_0(z \bar{\xi} / |\xi|)\|_{F_0} \leq \|f\|_{m_0}$$

and

$$f(z) \log(1/(1 - z \bar{\xi} / |\xi|)) = \int_T \log \left(\frac{1}{1 - \bar{\varsigma} z} \right) d\mu_\xi(\varsigma),$$

where $\mu_\xi \in M$, $\|\mu_\xi\| \leq \|f\|_{m_0}$.

Then

$$f'(z) \log(1/(1 - z \bar{\xi} / |\xi|)) = \int_T \frac{\bar{\varsigma}}{1 - \bar{\varsigma} z} d\mu_\xi(\varsigma) - f(z) (1/(1 - z \bar{\xi} / |\xi|)) \cdot (\bar{\xi} / |\xi|).$$

This implies that

$$|f'(z)| \left| \log(1/(1 - z\bar{\xi}/|\xi|)) \right| (1-|z|) \leq \|f\|_{m_0} + \|f\|_{H^\infty} (1-|z|)(1/|1 - z\bar{\xi}/|\xi||).$$

Substituting $z = \xi$ we obtain *a*).

The proof of *b*) can be found in [3] or [2].

Corollary 1. *If $f \in m_0$, then for each $\varsigma \in T$ and $p \geq 1$ we have*

$$\int_0^1 \left((1-r) \log \frac{1}{1-r} \right)^{p-1} |f'(r\varsigma)|^p dr \leq \frac{\pi}{2} \|f\|_{m_1} (\|f\|_{m_0} + \|f\|_{H^\infty})^{p-1}.$$

The Proof it follows at once from *a*) and *b*).

Let $m(B)$ denote the family of all multipliers of the Bloch space B .

Corollary 2. $m_0 \subset (m_\alpha \cap m(B))$ for each $\alpha > 0$.

Proof. It follows at once from Theorem A and the assertion that $f \in m_0$ if and only if $f \in H^\infty$ and satisfies the condition *a*) [4].

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